Interaction of *N* solitons in the massive Thirring model and optical gap system: The complex Toda chain model

V. S. Shchesnovich*

Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag 7701 Rondebosch, South Africa

(Received 30 September 2001; published 8 April 2002)

Using the Karpman-Solov'ev quasiparticle approach for soliton-soliton interaction we show that the train propagation of *N* well-separated solitons of the massive Thirring model is described by the complex Toda chain with *N* nodes. For the optical gap system a generalized (nonintegrable) complex Toda chain is derived for description of the train propagation of well-separated gap solitons. These results are in favor of the recently proposed conjecture of universality of the complex Toda chain.

DOI: 10.1103/PhysRevE.65.046614 PACS number(s): 05.45.Yv, 42.65.Tg, 42.81.Dp

I. INTRODUCTION

Recently the complex Toda chain attracted much attention as a possible candidate for description of the pulse interactions in integrable and nonintegrable nonlinear evolution equations [1–10]. For instance, it was shown that the complex Toda chain describes the soliton train propagation for all the nonlinear evolution equations associated with the nonlinear Schrödinger (NLS) hierarchy [9]. Quite recently the complex Toda chain was derived for the modified NLS (MNLS) equation [10], an integrable generalization of the NLS equation, which is associated with the quadratic bundle.

The complex Toda chain is an integrable generalization of the well-known real Toda chain (see, for instance, Refs. [3,4]). In Refs. [3,4,7,10] the comparison of the complex Toda chain predictions with the numerical solutions of the NLS and MNLS equations has been performed and a good agreement has been established for various choices of the initial parameters of the solitons in the train.

It is noted that the complex Toda chain arises as an approximation of the evolution equations describing the interpulse interaction in the train comprised of well-separated solitons with nearly equal amplitudes and velocities. The exponent of the (negative) separation between the solitons serves as the small parameter for the asymptotic expansion and derivation of the complex Toda chain can be based either on the variational approach (see Ref. [8]) or on the adiabatic perturbation theory for solitons (see, for instance, Refs. [4,10]). However, as noted in Ref. [11] the variational approach should be used with care. The approach based on the adiabatic perturbation theory is equivalent to the Karpman-Solov'ev quasiparticle method for the two-soliton interactions [12]. This approach was developed in Refs. [3,4,6,9,10].

If the nonlinear partial differential equation (PDE) is not integrable but possesses stable soliton solutions, then the train propagation of solitons is described by a generalized (nonintegrable) complex Toda chain as it is pointed out in Ref. [8].

The complex Toda chain allows a rich class of asymptotic regimes of the soliton train propagation [3,6,7]: (i) asymptotically free propagation of solitons, (ii) *N* soliton bound states with the possibility of a quasiequidistant propagation, (iii) mixed asymptotic regimes when part of the solitons form bound state(s) and the rest separate from them, (iv) regimes corresponding to the degenerate and singular solutions of the complex Toda chain. The rich variety of dynamical regimes of the complex Toda chain indicates that it is a good candidate for analytical study of the soliton trains. Here we should point out that only few simple regimes are exhibited by the real Toda chain [13,14], thus it is essential to have the complex Toda chain in description of the soliton trains. Moreover, the phase space of the complex Toda chain (CTC) with *N* nodes is $4N$ dimensional, which is precisely the number of real parameters in the train of *N* solitons.

In the present paper we consider the *N*-soliton train propagation governed by two intimately related nonlinear PDEs, one of which is integrable and the other not: the massive Thirring model of the classical field theory [15,16] and the optical gap system [17–30].

For the massive Thirring model we show that the train propagation of well-separated solitons with nearly equal amplitudes and rapidities is governed by the complex Toda chain. Moreover, we derive a nonintegrable generalization of the complex Toda chain, which describes the train propagation of well-separated gap solitons with nearly equal amplitudes and velocities.

The gap soliton propagation through a grated optical fiber was manifested in recent experiments [31–34]. The results of Ref. [34] are of particular interest, there the *multiple gap soliton formation* was observed.

Localized solutions in nonlinear media with periodic band gaps have a great potential for technological applications. One of the most important band gap structures in optics is given by an optical fiber with periodic index grating along the axis. From the Floquet-Bloch theory of wave propagation in periodic structures it is known that there are forbidden frequency bands or band gaps for linear waves. On the other hand, nonlinear wave propagation in such structures is possible for the central frequency of the wave packet lying in the band gap. Such nonlinear wave is usually called gap soliton.

The optical gap system was derived within the coupled

*On leave from The Division for Optical Problems in Information Technologies, National Academy of Sciences of Belarus, Kuprevich St. 1 block 2, Minsk-41, 220141, Belarus; electronic address: valery@optoinform.bas-net.by

mode approach for nonlinear wave propagation in optical fibers with grating (see, for instance, Ref. [28]). It reads

$$
\begin{aligned}
&i(E_{1T}-E_{1X})+E_2+(|E_2|^2+\rho|E_1|^2)E_1=0,\\
&i(E_{2T}+E_{2X})+E_1+(|E_1|^2+\rho|E_2|^2)E_2=0,
\end{aligned} \tag{1}
$$

where E_1 and E_2 are the slowly varying envelopes of two counterpropagating waves coupled through the Bragg scattering induced by the grating (the linear cross-coupling terms), the nonlinear terms account for the self- and cross-phase modulation effects. The parameter ρ ($\rho>0$) at the self-phase modulation term may range up to infinity [35], in which case the optical gap system models dynamics in the nonlinear dual-core asymmetric coupler [27]. Setting $\rho=0$ in the system (1) one obtains the massive Thirring model.

The gap solitons in optical fibers were studied theoretically in many works [17–28] (see also the latest review Ref. [29]). Recently, the relevance of the system (1) for description of the optical gap solitons was analytically and numerically validated [30]. The general family of the gap solitons was derived in Ref. [19] using the similarity with the massive Thirring model. After that, the N-soliton solutions for the optical gap system were analytically studied via similar approach in Ref. [36]. Recently, it was shown that the gap soliton becomes unstable when its amplitude grows above some fixed value [37,38].

The solitonlike solutions, which are similar to the gap solitons, were found in nonlinear diatomic lattices [39–41] and in the quadratic ($\chi^{(2)}$) materials with a spatially periodic linear susceptibility (grating) [42–49]. Also, it was shown [41,44] that, at some limit, the equations governing nonlinear wave propagation in quadratic media with grating and in diatomic lattices are similar to the system (1) though the underlying physics is different.

In the following section, Sec. II, we state the main results of the paper on the soliton train propagation for the massive Thirring model and optical gap system. The details of the derivation are placed in the following sections: Sec. III for the massive Thirring model and Sec. IV for the optical gap system. The last section contains discussion of the results and suggestions for further work.

II. MAIN RESULTS

Before formulating the main results we would like to recall some facts about the models under study. Let us begin with the massive Thirring model of the classical field theory [15,16],

$$
\begin{aligned}
&i(v_t-v_x)+u+|u|^2v=0,\\
&i(u_t+u_x)+v+|v|^2u=0,
\end{aligned} \tag{2}
$$

where u and v are complex variables, t and x are the time and space coordinates, respectively. The system (2) is Lorentz invariant,

$$
x\to\frac{x-\tanh(y)t}{(1-\tanh^2 y)^{1/2}},\quad t\to\frac{t-\tanh(y)x}{(1-\tanh^2 y)^{1/2}},
$$

with u and v transforming as components of the Lorentz spinor,

$$
u\to e^{-y/2}u,\quad v\to e^{y/2}v.
$$

In the relativistic kinematics the parameter y is called "rapidity." (Rapidities of two consecutive Lorentz transformations simply add together.)

The massive Thirring model (MTM) is integrable by the inverse scattering transform method [16]. For instance, its one-soliton solution can be written as

$$
\begin{aligned}
v&=-\frac{i\sin(2\vartheta)\exp(-y/2+i\Theta)}{\cosh(z-i\vartheta)},\\
u&=\frac{i\sin(2\vartheta)\exp(y/2+i\Theta)}{\cosh(z+i\vartheta)},
\end{aligned} \tag{3}
$$

where $0<\vartheta<\pi/2$ and

$$
\begin{aligned}
z&=\sin(2\vartheta)\cosh(y)[x_o(t)-x],\\
\Theta&=-\cot(2\vartheta)\tanh(y)z+\delta(t).
\end{aligned} \tag{4}
$$

The soliton has four independent real parameters: ϑ, y, x_o, and δ. The first two give the soliton amplitude and rapidity, while the rest two are the soliton position and central phase (the phase at $x=x_o$), respectively. The position and phase parameters depend on time,

$$
\frac{dx_o}{dt}=\tanh y,\quad \frac{d\delta}{dt}=-\cos(2\vartheta)\mathrm{sech}\, y. \tag{5}
$$

The first equation defines the soliton velocity, $V=\tanh y$.

By a suitable Lorentz transformation the rapidity of a Thirring soliton can be put equal to zero and the soliton solution (3) reduces to the quiescent soliton of the massive Thirring model.

The following *ansatz* is called the soliton train:

$$
\begin{aligned}
v&=\sum_{\alpha=1}^{N}-\frac{i\sin(2\vartheta_\alpha)\exp(-y_\alpha/2+i\Theta_\alpha)}{\cosh(z_\alpha-i\vartheta_\alpha)},\\
u&=\sum_{\alpha=1}^{N}\frac{i\sin(2\vartheta_\alpha)\exp(y_\alpha/2+i\Theta_\alpha)}{\cosh(z_\alpha+i\vartheta_\alpha)},
\end{aligned} \tag{6}
$$

where z_α and Θ_α are given by formulas similar to Eqs. (4) and (5). It should be stressed that, for each soliton, all four soliton parameters in formula (6) are considered to be t dependent.

The CTC for the MTM soliton train. Assume that the N-soliton train given by Eq. (6) consists of well-separated pulses with nearly equal amplitudes ϑ_α and rapidities y_α, numerated by $\alpha=1,\dots,N$ in such a way that $x_{\alpha+1}-x_\alpha>0$ (here and below x_α denotes the position parameter "x_o" for the αth soliton). Mathematically, these conditions are expressed as

$$|\vartheta_\alpha-\bar{\vartheta}|\ll\bar{\vartheta},\quad |y_\alpha-\bar{y}|\ll 1,\quad |x_\alpha-x_{\alpha\pm 1}|\gg 1,$$
$$|\sin(2\vartheta_\alpha)\cosh y_\alpha-\sin(2\bar{\vartheta})\cosh\bar{y}||x_\alpha-x_{\alpha\pm1}|\ll 1. \tag{7}$$

Here (and throughout the paper) $\bar{\vartheta}$ and $\bar{y}$ denote the averages,

$$\bar{\vartheta}=\frac{1}{N}\sum_{\alpha=1}^{N}\vartheta_\alpha,\quad \bar{y}=\frac{1}{N}\sum_{\alpha=1}^{N}y_\alpha. \tag{8}$$

Define the following new variables: a modified time

$$\tau=\sin(2\bar{\vartheta})\operatorname{sech}(\bar{y})t, \tag{9}$$

an average phase

$$\bar{\delta}=-\cos(2\bar{\vartheta})\operatorname{sech}(\bar{y})t, \tag{10}$$

and the following complex variable for each soliton:

$$q_\alpha=-\sin(2\bar{\vartheta})\cosh(\bar{y})x_\alpha-i[\delta_\alpha-\bar{\delta}-\cos(2\bar{\vartheta})\sinh(\bar{y})x_\alpha+\alpha\pi]+2\alpha\ln[2\sin(2\bar{\vartheta})]. \tag{11}$$

Then in the first order of the soliton overlap parameter ϵ,

$$\epsilon\simeq\exp\{-|\sin(2\vartheta_\alpha)\cosh(y_\alpha)x_\alpha-\sin(2\vartheta_{\alpha\pm1})\cosh(y_{\alpha\pm1})x_{\alpha\pm1}|\}, \tag{12}$$

the following two statements are claimed: (1) the average values $\bar{\vartheta}$ and $\bar{y}$ do not depend on t; (2) evolution of the quantities q_α, $\alpha=1,\dots,N$, is given by the complex Toda chain with N nodes,

$$\frac{d^2q_\alpha}{d\tau^2}=e^{q_{\alpha+1}-q_\alpha}-e^{q_\alpha-q_{\alpha-1}},\quad \alpha=1,\dots,N, \tag{13}$$

where $\mathrm{Re}\{q_0\}=\infty$ and $\mathrm{Re}\{q_{N+1}\}=-\infty$ [i.e., $x_0=-\infty$ and $x_{N+1}=\infty$, see Eq. (11)].

The set of inequalities (7) is similar to the inequalities for the NLS soliton train in Ref. [4].

Now we will formulate similar result for the train propagation of pulses governed by the optical gap system [28],

$$i(E_{1t}-E_{1x})+E_2+(|E_2|^2+\rho|E_1|^2)E_1=0,$$
$$i(E_{2t}+E_{2x})+E_1+(|E_1|^2+\rho|E_2|^2)E_2=0. \tag{14}$$

The soliton solution of the optical gap system (14) moving with the velocity $V=\tanh y_o$ reads [19]

$$\begin{pmatrix}E_1(x,t)\\E_2(x,t)\end{pmatrix}=\frac{e^{i\psi(x,t)}}{[1+\rho\cosh(2y_o)]^{1/2}}\begin{pmatrix}v(x,t)\\u(x,t)\end{pmatrix}, \tag{15}$$

where v and u have the form of a Thirring soliton, i.e., given by formulas (3)–(5) (with $y\to y_o$); the additional (nonlinear) phase ψ is

$$\psi=-\frac{2\rho\sinh(2y_o)}{1+\rho\cosh(2y_o)}\arctan(\tan\vartheta\tanh z), \tag{16}$$

with z as in Eq. (4).

Here it should be pointed out that the gap soliton becomes unstable when the soliton amplitude ϑ grows above certain threshold ($\vartheta_{thr}\approx\pi/4$, see Ref. [37]). This scenario is also possible for the train of gap solitons. This instability is the result of the soliton-radiation interaction and is beyond the scope of the adiabatic approach. However, being interested in *stable* gap solitons, one can impose the condition $\vartheta_\alpha<\vartheta_{thr}$, for all $\alpha=1,\dots,N$.

The *ansatz* we use for the train of *well-separated* gap solitons is given by application of the transformation (15) to the train of well-separated Thirring solitons (in this case $y_o=\bar{y}$). Due to the inequalities (7), the additional phase ψ_α of each soliton in the train can be approximated by formula (16) with $\vartheta=\bar{\vartheta}$ and $z_\alpha=\sin(2\bar{\vartheta})\cosh(\bar{y})(x_\alpha-x)$.

The generalized CTC for the train of gap solitons. Assume that the train of N-gap solitons consists of well-separated pulses with nearly equal amplitudes ϑ_α and rapidities y_α numerated by $\alpha=1,\dots,N$ in such a way that $x_{\alpha+1}-x_\alpha>0$ and that the conditions (7) are satisfied. Associate the following variable with each gap soliton:

$$Q_\alpha=-\sin(2\bar{\vartheta})\cosh(\bar{y})x_\alpha-i\{\delta_\alpha-\bar{\delta}-[\cos(2\bar{\vartheta})-\mu\sin(2\bar{\vartheta})(y_\alpha-\bar{y})]\sinh(\bar{y})x_\alpha+\alpha\pi\}+2\alpha\ln[2\sin(2\bar{\vartheta})], \tag{17}$$

where

$$\mu=\frac{4\rho\tanh(2\bar{y})}{\rho+\operatorname{sech}(2\bar{y})}\bar{\vartheta}.$$

Define the modified time τ and average phase $\bar{\delta}$ as in formulas (9) and (10). Then, in the first order of the soliton overlap parameter ϵ Eq. (12), the following is claimed: (1) the average values $\bar{\vartheta}$ and $\bar{y}$ do not depend on t; (2) evolution of the quantities Q_α, $\alpha=1,\dots,N$, is given by the following generalized complex Toda chain with N nodes,

$$\frac{d^2Q_\alpha}{d\tau^2}=(1+A_\rho)(e^{Q_{\alpha+1}-Q_\alpha}-e^{Q_\alpha-Q_{\alpha-1}})+B_\rho(e^{Q^*_{\alpha+1}-Q^*_\alpha}-e^{Q^*_\alpha-Q^*_{\alpha-1}}), \tag{18}$$

where $\mathrm{Re}\{Q_0\}=\infty$ and $\mathrm{Re}\{Q_{N+1}\}=-\infty$.

Equation (18) is valid for *arbitrary* values of the self-phase modulation parameter ρ.

Here A_ρ and B_ρ are ρ-dependent coefficients

$$A_\rho=\tfrac{1}{2}\{\nu-\kappa\mu+i[\kappa(1+\nu)+\mu]\},$$
$$B_\rho=\tfrac{1}{2}\{\nu+\kappa\mu-i[\kappa(1+\nu)-\mu]\}, \tag{19}$$

with

$$\kappa=\frac{\rho\tanh(2\overline{y})}{\rho+\mathrm{sech}(2\overline{y})}\,\frac{4\overline{\vartheta}-\sin(4\vartheta)}{\sin^2(2\vartheta)},$$

$$\nu=\frac{4\rho(2\overline{\vartheta}\cot(2\overline{\vartheta})-1)}{\rho+\mathrm{sech}(2\overline{y})}.$$

Setting $\rho=0$ in Eqs. (17) and (18) one obtains the complex Toda chain for the soliton train of the massive Thirring model.

III. THE CTC FOR THE MTM SOLITON TRAIN

We will use the adiabatic perturbation theory for derivation of the complex Toda chain. Recently the perturbation theory based on the Riemann-Hilbert problem was developed for the solitons of the massive Thirring model [50]. For instance, we have derived the following evolution equations for the soliton parameters in the adiabatic approximation,

$$\frac{d\vartheta}{dt}=-\frac{1}{2\cosh y}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\cosh(2z)+\cos(2\vartheta)}\,\mathrm{Re}\{e^{z}[r_{\perp}(k_1,z)+r_{\perp}^{*}(k_1^{*},-z)]\},\tag{20}$$

$$\frac{dy}{dt}=\frac{1}{\cosh y}\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\cosh(2z)+\cos(2\vartheta)}\,\mathrm{Im}\{e^{z}[r_{\perp}(k_1,z)+r_{\perp}^{*}(k_1^{*},-z)]-2r_{\parallel}(z)\},\tag{21}$$

$$\frac{dx_o}{dt}=\tanh y-\frac{\mathrm{Re}\{J\}}{2\sin^2(2\vartheta)\cosh^2 y},\tag{22}$$

$$\frac{d\delta}{dt}=-\cos(2\vartheta)\mathrm{sech}\,y-\frac{\mathrm{Im}\{J\}+\tanh y\cot(2\vartheta)\mathrm{Re}\{J\}}{2\sin(2\vartheta)\cosh y}.\tag{23}$$

Here

$$J=\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\cosh(2z)+\cos(2\vartheta)}(2\{2z[\cos(2\vartheta)+i\sin(2\vartheta)\tanh y]+\sinh(2z)\}r_{\parallel}(z)+\{e^{z}[\cos(2\vartheta)(1-2z)-2iz\sin(2\vartheta)\tanh y]+e^{-z}\}\times[r_{\perp}(k_1,z)-r_{\perp}^{*}(k_1^{*},-z)]).\tag{24}$$

The functions $r_{\perp}$ and $r_{\parallel}$ are given by

$$r_{\perp}(k_1,z,t)=\frac{ie^{-i\Theta}}{2}\left(k_1\frac{\delta v}{\delta t}-k_1^{-1}\frac{\delta u}{\delta t}\right),\tag{25}$$

$$r_{\parallel}(z,t)=\frac{i}{4}\left(\frac{\delta|v|^2}{\delta t}-\frac{\delta|u|^2}{\delta t}\right),\tag{26}$$

where $k_1=-\exp\{-y/2-i\vartheta\}$. The "variational" derivatives denote fictitious evolution of the u and v as if under the action of the perturbation only [in other words, either $i\delta u/\delta t$ or $i\delta v/\delta t$ is nothing but the perturbation added to the respective right-hand side of the massive Thirring model (2)]. Equations (20)–(26) can also be derived via the perturbation theory for the Thirring solitons developed in Ref. [51].

From the point of view of the inverse scattering transform method the *ansatz* (6) contains not only the N-soliton solution but also the contribution of radiation as well. However, due to large separations between the solitons in the train, the radiation component is negligible. Thus, we can use the adiabatic perturbation theory for derivation of evolution equations for the soliton parameters ϑ_α, y_α, x_α, and δ_α. For the same reason, it is sufficient to consider the interaction between the neighboring pulses only (detailed discussion can be found in Ref. [4]).

First, one should compute the perturbation functions defined in Eqs. (25) and (26). Substitution of the *ansatz* (6) into the massive Thirring model (2) and expansion of the cubic terms leads to the following formulas for the perturbation-induced evolutions (we consider the interaction between the neighboring pulses only):

$$\begin{aligned} i\frac{\delta v_\alpha}{\delta t}&=-\sum_{\beta=\alpha\mp1}(|u_\alpha|^2v_\beta+2\,\mathrm{Re}\{u_\alpha u_\beta^{*}\}v_\alpha),\\ i\frac{\delta u_\alpha}{\delta t}&=-\sum_{\beta=\alpha\mp1}(|v_\alpha|^2u_\beta+2\,\mathrm{Re}\{v_\alpha v_\beta^{*}\}u_\alpha),\end{aligned}\tag{27}$$

and, consequently,

$$\begin{aligned} i\frac{\delta|v_\alpha|^2}{\delta t}&=\sum_{\beta=\alpha\mp1}2i|u_\alpha|^2\,\mathrm{Im}\{v_\alpha v_\beta^{*}\},\\ i\frac{\delta|u_\alpha|^2}{\delta t}&=\sum_{\beta=\alpha\mp1}2i|v_\alpha|^2\,\mathrm{Im}\{u_\alpha u_\beta^{*}\}.\end{aligned}\tag{28}$$

Before giving the perturbation-induced evolution of the soliton parameters some remarks must be made on the details of the approximation due to the inequalities (7). The right-hand sides (rhs's) in Eq. (27) contain the small parameter ϵ [Eq. (12)]. We consider the soliton-soliton interaction in the first order with respect to ϵ. Hence, due to the presence of the small parameter, the differences between the αth soliton amplitude and rapidity and the average values of these quantities [given by Eq. (8)] are negligible in the terms accounting for the intersoliton interaction.

Substitution of Eqs. (27) and (28) into Eqs. (25) and (26) and the result into Eqs. (20) and (21) gives the following equations for the amplitudes and rapidities:

$$\frac{d\vartheta_\alpha}{dt}=\sum_{\beta=\alpha\mp1}\frac{2\sin^3(2\overline{\vartheta})}{\cosh\overline{y}}e^{-|\Delta_{\alpha\beta}|}\sin\Psi_{\alpha\beta},\tag{29}$$

$$\frac{dy_\alpha}{dt}=\sum_{\beta=\alpha\mp1}\frac{4\,\mathrm{sgn}(\Delta_{\alpha\beta})\sin^3(2\overline{\vartheta})}{\cosh\overline{y}}e^{-|\Delta_{\alpha\beta}|}\cos\Psi_{\alpha\beta},\tag{30}$$

where

$$\Delta_{\alpha\beta}=\sin(2\overline{\vartheta})\cosh(\overline{y})(x_\beta-x_\alpha),$$

$$\Psi_{\alpha\beta}=\delta_\alpha-\delta_\beta-\cos(2\overline{\vartheta})\sinh(\overline{y})(x_\alpha-x_\beta).$$

Here $\exp(-|\Delta_{\alpha\beta}|)=O(\epsilon)$. It can be easily verified that Eqs. (29) and (30) do not affect the average amplitude $\overline{\vartheta}$ and rapidity $\overline{y}$.

The evolution equations for x_α and δ_α [see Eqs. (22) and (23)] are comprised of two addends, which account for the unperturbed and perturbation-induced evolution (the latter contain the soliton overlap parameter ϵ). Hence, one can neglect the terms accounting for the perturbation-induced evolution of these parameters as compared to their unperturbed evolution,

$$\frac{dx_\alpha}{dt}=\tanh y_\alpha, \quad \frac{d\delta_\alpha}{dt}=-\cos(2\vartheta_\alpha)\mathrm{sech}\, y_\alpha. \tag{31}$$

Now everything is ready for derivation of the complex Toda chain. Let us differentiate the following quantity $-\Delta_{\alpha\beta}+i\Psi_{\alpha\beta}$ ($\beta=\alpha\pm1$):

$$\frac{d}{dt}(-\Delta_{\alpha\beta}+i\Psi_{\alpha\beta})=\sin(2\overline{\vartheta})\mathrm{sech}(\overline{y})[(y_\alpha+2i\vartheta_\alpha)-(y_\beta+2i\vartheta_\beta)],$$

where the second-order terms in $(\vartheta_\alpha-\overline{\vartheta})$ and $(y_\alpha-\overline{y})$ are dropped. On the other hand, from Eqs. (29) and (30) one derives

$$\frac{d}{dt}(y_\alpha+2i\vartheta_\alpha)=\sum_{\beta=\alpha\mp1}\frac{4\ \mathrm{sgn}(\Delta_{\alpha\beta})\sin^3(2\overline{\vartheta})}{\cosh\overline{y}}\times\exp\{\mathrm{sgn}(\Delta_{\alpha\beta})(-\Delta_{\alpha\beta}+i\Psi_{\alpha\beta})\},$$

or using the numeration $x_{\alpha+1}-x_\alpha>0$, i.e., $\mathrm{sgn}(\Delta_{\alpha,\alpha+1})>0$, for the solitons in the train,

$$\frac{d}{dt}(y_\alpha+2i\vartheta_\alpha)=\frac{4\sin^3(2\overline{\vartheta})}{\cosh\overline{y}}(\exp\{-\Delta_{\alpha\alpha+1}+i\Psi_{\alpha\alpha+1}\}-\exp\{\Delta_{\alpha\alpha-1}-i\Psi_{\alpha\alpha-1}\}). \tag{32}$$

Introduce an average phase

$$\overline{\delta}=-\cos(2\overline{\vartheta})\mathrm{sech}(\overline{y})t,$$

and the following complex variable associated with each soliton:

$$q_\alpha=-\sin(2\overline{\vartheta})\cosh(\overline{y})x_\alpha-i[\delta_\alpha-\overline{\delta}+\alpha\pi-\cos(2\overline{\vartheta})\sinh(\overline{y})x_\alpha]+2\alpha\ln[2\ \sin(2\overline{\vartheta})].$$

(The average phase in the formula for q_α eliminates the constant phase gradient.) Then

$$\exp[\pm(q_{\alpha\pm1}-q_\alpha)]=-4\sin^2(2\overline{\vartheta})\times\exp(\mp\Delta_{\alpha\alpha\pm1}\pm i\Psi_{\alpha\alpha\pm1}). \tag{33}$$

Differentiating q_α and neglecting the second-order terms, we get

$$\frac{dq_\alpha}{dt}=-\sin(2\overline{\vartheta})\cosh\overline{y}\tanh y_\alpha-i[\sin(2\overline{\vartheta})\mathrm{sech}(\overline{y})\times(2\vartheta_\alpha-2\overline{\vartheta})+\cos(2\overline{\vartheta})\mathrm{sech}\,\overline{y}\tanh(\overline{y})(y_\alpha-\overline{y})-\cos(2\overline{\vartheta})\sinh\overline{y}\tanh y_\alpha]$$

(here the average phase $\overline{\delta}$ plays an important role for the expansion over $\vartheta_\alpha-\overline{\vartheta}$). The second differentiation (with removal of the second-order terms) gives

$$\frac{d^2q_\alpha}{dt^2}=-\sin(2\overline{\vartheta})\mathrm{sech}(\overline{y})\left(\frac{dy_\alpha}{dt}+2i\frac{d\vartheta_\alpha}{dt}\right). \tag{34}$$

At the same time, as it follows from Eqs. (32) and (33) that

$$\frac{d}{dt}[-y_\alpha-i(2\vartheta_\alpha-2\overline{\vartheta})]=\sin(2\overline{\vartheta})\mathrm{sech}(\overline{y})\times(e^{q_{\alpha+1}-q_\alpha}-e^{q_\alpha-q_{\alpha-1}}). \tag{35}$$

Therefore, what is left is to introduce a new time variable

$$\tau=\sin(2\overline{\vartheta})\mathrm{sech}(\overline{y})t.$$

Then Eqs. (34) and (35) give the complex Toda chain for the train of Thirring solitons,

$$\frac{d^2q_\alpha}{d\tau^2}=e^{q_{\alpha+1}-q_\alpha}-e^{q_\alpha-q_{\alpha-1}}, \quad \alpha=1,\ldots,N.$$

Here it is assumed that $\mathrm{Re}\{q_0\}=\infty$ and $\mathrm{Re}\{q_{N+1}\}=-\infty$ (i.e., $x_0=-\infty$ and $x_{N+1}=\infty$).

IV. THE GENERALIZED CTC FOR THE TRAIN OF GAP SOLITONS

In the derivation of the complex Toda chain for the optical gap system we will use a one-to-one mapping, found recently in Ref. [50], between the optical gap system (1) and the following generalization of the massive Thirring model, the γ system for short,

$$\begin{aligned} &i(\mathcal{V}_t-\mathcal{V}_x)+\mathcal{U}+|\mathcal{U}|^2\mathcal{V}+\gamma_-(|\mathcal{V}|^2-|\mathcal{U}|^2)\mathcal{V}=0,\\ &i(\mathcal{U}_t+\mathcal{U}_x)+\mathcal{V}+|\mathcal{V}|^2\mathcal{U}+\gamma_+(|\mathcal{U}|^2-|\mathcal{V}|^2)\mathcal{U}=0.\end{aligned} \tag{36}$$

The transformation relating the two systems is as follows. Let $\mathcal{U}(x,t)$ and $\mathcal{V}(x,t)$ be a solution of the γ system (36) with the following $\gamma_\pm$:

$$\gamma_\pm=\frac{\rho e^{\pm2y_o}}{1+\rho\cosh(2y_o)}, \tag{37}$$

then

$$\begin{pmatrix} E_1(X,T) \\ E_2(X,T) \end{pmatrix} = \frac{e^{i\psi(x,t)}}{[1+\rho\cosh(2y_o)]^{1/2}} \begin{pmatrix} e^{-y_o/2}\mathcal{V}(x,t) \\ e^{y_o/2}\mathcal{U}(x,t) \end{pmatrix}, \tag{38}$$

where

$$x = \frac{X-\tanh(y_o)T}{[1-\tanh^2(y_o)]^{1/2}}, \quad t = \frac{T-\tanh(y_o)X}{[1-\tanh^2(y_o)]^{1/2}}, \tag{39}$$

is a solution to the optical gap system (1) with the phase ψ given by the following system of equations [in the light-cone variables $\eta=(t+x)/2$ and $\xi=(t-x)/2$]:

$$\frac{\partial\psi}{\partial\eta} = \frac{1}{2}(\gamma_+ - \gamma_-)|\mathcal{V}|^2, \quad \frac{\partial\psi}{\partial\xi} = -\frac{1}{2}(\gamma_+ - \gamma_-)|\mathcal{U}|^2. \tag{40}$$

(Note that the conservation of the number of particles, i.e.,

$$\frac{\partial}{\partial\xi}|\mathcal{V}|^2 + \frac{\partial}{\partial\eta}|\mathcal{U}|^2 = 0,$$

ensures the compatibility of the equations for the phase ψ.)

Note that the coordinates are related via a Lorentz transformation. The mapping (38) can be verified by direct substitution into Eq. (1) via simple calculations with the use of Eqs. (37), (39), and (40).

The presented mapping is valid for *arbitrary* solutions of the optical gap system. However, the importance of the mapping (37)–(40) stems from the fact that by choosing the *quiescent* Thirring soliton,

$$\mathcal{V} = -\frac{i\sin(2\vartheta)e^{i\delta}}{\cosh(z-i\vartheta)}, \quad \mathcal{U} = \frac{i\sin(2\vartheta)e^{i\delta}}{\cosh(z+i\vartheta)},$$

$$z = -\sin(2\vartheta)(x-x_o), \quad \frac{d\delta}{dt} = -\cos(2\vartheta),$$

which is a solution to the system (36) due to $|\mathcal{V}|=|\mathcal{U}|$, one can recover the optical gap soliton *moving with any given velocity* $V=\tanh y_o$. In this case one obtains formula (16) for the additional phase ψ. Although the optical gap system is not Lorentz invariant, still it makes sense to call y_o "rapidity" of the gap soliton due to the transformation (37)–(40).

A train of N well-separated gap solitons moving with arbitrary average rapidity y_o can always be represented via the transformation (37)–(40) as a train of N well-separated *almost quiescent* Thirring solitons. Indeed, application of the mapping (37)–(40) to Eq. (6) with $\mathcal{V}=v$ and $\mathcal{U}=u$, under the conditions $|y_\alpha|\ll 1$ and $\overline{y}=0$, yields the train of N well-separated gap solitons with nearly equal amplitudes and rapidities, where the *average* rapidity is equal to the given *arbitrary* value y_o. Moreover, if one neglects the terms of order $O(\epsilon)$, then the additional phase ψ_α of each gap soliton in the train is determined by Eq. (16) with evident changes: $\vartheta\to\overline{\vartheta}$, $z\to z_\alpha$, and $x_o\to x_\alpha$.

Convenience of the γ system (36) for the analytical study of gap solitons is based on the two following facts. First, the quiescent Thirring soliton satisfies $|\mathcal{U}|=|\mathcal{V}|$. Hence, the last terms in the γ system are small if the solution under study is close to the quiescent Thirring soliton, or, in terms of the optical gap system, the solution is close to the gap soliton. Second, the parameters $\gamma_\pm$ are bounded for all values of y_o and ρ (including $\rho=\infty$). Therefore, the use of the equivalent γ system allows one to apply the perturbation theory developed for the train of almost quiescent Thirring solitons to the train of gap solitons for *arbitrary* values of the self-phase modulation parameter ρ.

Hence, derivation of the complex Toda chain for the gap soliton train can be done in much the same way as the derivation of the complex Toda chain for the train of Thirring solitons (more precisely, almost quiescent Thirring solitons). The only difference is that there are additional small perturbations given by the terms with $\gamma_\pm$ in Eq. (36), the γ terms for short. Let us first calculate their contribution to the evolution of the soliton parameters and then calculate evolution of q_α, defined in a similar way as in the preceding section, with account of these terms as well.

Below we will take into account that the gap soliton train is transformed by the mapping (37)–(40) with $y_o=\overline{y}$ to the train of almost quiescent Thirring solitons (in the variables $\mathcal{V}$ and $\mathcal{U}$). For instance, the latter train has $\overline{y}=0$ and $|y_\alpha|\ll 1$. Consider the αth soliton in such train. From Eq. (36) one gets, expanding the cubic terms,

$$i\frac{\delta\mathcal{V}_\alpha}{\delta t} = -\sum_{\beta=\alpha\mp1}(|\mathcal{U}_\alpha|^2\mathcal{V}_\beta + 2\,\mathrm{Re}\{\mathcal{U}_\alpha\mathcal{U}_\beta^*\}\mathcal{V}_\alpha) - \gamma_-(|\mathcal{V}_\alpha|^2 - |\mathcal{U}_\alpha|^2)\mathcal{V}_\alpha - \sum_{\beta=\alpha\mp1}\gamma_-(2|\mathcal{V}_\alpha|^2\mathcal{V}_\beta + \mathcal{V}_\alpha^2\mathcal{V}_\beta^* - |\mathcal{U}_\alpha|^2\mathcal{V}_\beta - 2\,\mathrm{Re}\{\mathcal{U}_\alpha\mathcal{U}_\beta^*\}\mathcal{V}_\alpha).$$

Let us separate the rhs into two parts. The first part is just the same as in the case of the massive Thirring solitons, while the second, given by the following formula,

$$i\left(\frac{\delta\mathcal{V}_\alpha}{\delta t}\right)_\gamma = -\gamma_-(|\mathcal{V}_\alpha|^2 - |\mathcal{U}_\alpha|^2)\mathcal{V}_\alpha - \sum_{\beta=\alpha\mp1}\gamma_-(2|\mathcal{V}_\alpha|^2\mathcal{V}_\beta + \mathcal{V}_\alpha^2\mathcal{V}_\beta^* - |\mathcal{U}_\alpha|^2\mathcal{V}_\beta - 2\,\mathrm{Re}\{\mathcal{U}_\alpha\mathcal{U}_\beta^*\}\mathcal{V}_\alpha), \tag{41}$$

is due to the γ perturbation and is specific for the gap soliton train only. I will consider, in detail, only the γ perturbation since the first part is accounted for in just the same way as the train of almost quiescent Thirring solitons with the same resulting formulas.

In formula (41) the first term on the rhs is due to the self-interaction of the gap soliton [due to the γ terms in Eq. (36), if the rapidity $y_\alpha\neq 0$], the rest account for the intersoliton interaction in the train. The latter terms contain the small parameter ϵ defined in Eq. (12). We will use the same approximation that has been used for the derivation of the complex Toda chain for the massive Thirring model. Additionally one can neglect the difference between the modules of $\mathcal{V}_\alpha$ and $\mathcal{U}_\alpha$ when calculating the contribution from the intersoliton interaction terms. This is because the intersoliton interaction terms already contain the small parameter ϵ and $|y_\alpha|\ll 1$, thus one can put $y_\alpha=0$ there. Then, the "variational" derivatives simplify considerably,

$$i\left(\frac{\delta\mathcal{V}_\alpha}{\delta t}\right)_\gamma=-\gamma_-(|\mathcal{V}_\alpha|^2-|\mathcal{U}_\alpha|^2)\mathcal{V}_\alpha-\sum_{\beta=\alpha\mp 1}4\gamma_-\sin(\Psi_{\alpha\beta})\mathrm{Im}\{\mathcal{V}_{o\alpha}\mathcal{U}_{o\beta}\}\mathcal{V}_\alpha, \tag{42}$$

$$i\left(\frac{\delta\mathcal{U}_\alpha}{\delta t}\right)_\gamma=-\gamma_+(|\mathcal{U}_\alpha|^2-|\mathcal{V}_\alpha|^2)\mathcal{V}_\alpha-\sum_{\beta=\alpha\mp 1}4\gamma_+\sin(\Psi_{\alpha\beta})\mathrm{Im}\{\mathcal{U}_{o\alpha}\mathcal{V}_{o\beta}\}\mathcal{U}_\alpha, \tag{43}$$

where $\mathcal{V}_{o\alpha}=\exp\{-i\delta_\alpha\}\mathcal{V}_\alpha$, $\mathcal{U}_{o\alpha}=\exp\{-i\delta_\alpha\}\mathcal{U}_\alpha$, and $\Psi_{\alpha\beta}=\delta_\alpha-\delta_\beta$. Formula (43) obtains from Eq. (42) by the evident substitution $\mathcal{V}\rightarrow\mathcal{U}$, $\mathcal{U}\rightarrow\mathcal{V}$, and $\gamma_-\rightarrow\gamma_+$. Note that from formulas (42) and (43) it follows that

$$\left(\frac{\delta|\mathcal{V}_\alpha|^2}{\delta t}\right)_\gamma=0,\quad \left(\frac{\delta|\mathcal{U}_\alpha|^2}{\delta t}\right)_\gamma=0.$$

Consider first the contribution to evolution of the soliton parameters coming from the intersoliton interaction γ terms, i.e., the first terms on the rhs's in formulas (42) and (43). First, the perturbation functions given in Eqs. (25) and (26) must be calculated. The intersoliton interaction terms give the following contributions to the necessary functions: $r_\|(z_\alpha)=0$ and

$$r_\perp(k_\alpha,z_\alpha)+r_\perp^*(k_\alpha^*,-z_\alpha)=-\sum_{\beta=\alpha\mp 1}8i\frac{\sin^4(2\bar{\vartheta})e^{-|\Delta_{\alpha\beta}|}\sin(\Psi_{\alpha\beta})}{[\cosh(2z_\alpha)+\cos(2\bar{\vartheta})]^2}\sinh(2z_\alpha)\times[(\gamma_+-\gamma_-)e^{z_\alpha}+(\gamma_+e^{2i\bar{\vartheta}}-\gamma_-e^{-2i\bar{\vartheta}})e^{-z_\alpha}], \tag{44}$$

where $k_\alpha=-\exp\{-y_\alpha-i\vartheta_\alpha\}$ and $\Delta_{\alpha\beta}=\sin(2\bar{\vartheta})(x_\beta-x_\alpha)$. [On the rhs of Eq. (44) the terms of the second order in $\vartheta_\alpha-\bar{\vartheta}$ are neglected due to the small multiplier $\exp(-|\Delta_{\alpha\beta}|)=O(\epsilon)$.] Now, substitution of the expression (44) into Eqs. (20) and (21) leads to the following contributions to evolution equations for ϑ_α and y_α:

$$\left(\frac{d\vartheta_\alpha}{dt}\right)_\gamma=0,$$

$$\left(\frac{dy_\alpha}{dt}\right)_\gamma=-\sum_{\beta=\alpha\mp 1}4\kappa\sin^3(2\bar{\vartheta})e^{-|\Delta_{\alpha\beta}|}\sin\Psi_{\alpha\beta},$$

where

$$\kappa=\frac{\rho\tanh(2\bar{y})}{\rho+\mathrm{sech}(2\bar{y})}\,\frac{4\bar{\vartheta}-\sin(4\vartheta)}{\sin^2(2\vartheta)}.$$

What concerns the other two parameters x_α and δ_α, similarly as in Sec. III, the intersoliton interaction is of order $O(\epsilon)$ and its contribution to evolution of the soliton position and phase can be neglected as compared to their unperturbed evolution.

Now let us consider the contribution to evolution of the soliton parameters coming from the self-interaction γ terms, i.e., the first terms in Eqs. (42) and (43). We get the following contributions to the functions in Eqs. (25) and (26):

$$r_\|(z_\alpha)=0,\quad r_\perp(k_\alpha,z_\alpha)+r_\perp^*(k_\alpha^*,-z_\alpha)=0, \tag{45}$$

$$r_\perp(k_\alpha,z_\alpha)-r_\perp^*(k_\alpha^*,-z_\alpha)=\frac{4i\sin^3(2\vartheta_\alpha)y_\alpha}{[\cosh(2z_\alpha)+\cos(2\vartheta_\alpha)]^2}\times[(\gamma_--\gamma_+)e^{z_\alpha}+(\gamma_-e^{-2i\vartheta_\alpha}-\gamma_+e^{2i\vartheta_\alpha})e^{-z_\alpha}] \tag{46}$$

(here it is taken into account that terms of the second order in y_α are negligible). From Eq. (45) it follows that the contributions from the self-interaction γ terms to evolution equations for the amplitude ϑ_α and rapidity y_α vanish, while substitution of Eqs. (45) and (46) into Eq. (24) gives due to $|y_\alpha|\ll 1$,

$$J_\alpha=2\{(\gamma_-+\gamma_+)[2\sin^2(2\vartheta_\alpha)-2\vartheta_\alpha\sin(4\vartheta_\alpha)]+i(\gamma_--\gamma_+)2\vartheta_\alpha\sin^2(2\vartheta_\alpha)\}y_\alpha.$$

Thus the contributions from the self-interaction γ terms to evolution of x_α and δ_α are determined by the following coefficients:

$$\nu_\alpha=-\frac{\mathrm{Re}\{J_\alpha\}}{2\sin^2(2\vartheta_\alpha)y_\alpha}=(\gamma_-+\gamma_+)[4\vartheta_\alpha\cot(2\vartheta_\alpha)-2],$$

$$\mu_\alpha=-\frac{\mathrm{Im}\{J_\alpha\}}{2\sin^2(2\vartheta_\alpha)y_\alpha}=(\gamma_+-\gamma_-)2\vartheta_\alpha.$$

We can take just the average values of these coefficients (denoted below as ν and μ), because the following combinations $\nu_\alpha y_\alpha$ and $\mu_\alpha y_\alpha$ will enter the evolution equations for the soliton parameters and $|y_\alpha|\ll 1$. In other words, one can throw away the terms of the second order in $\vartheta_\alpha-\bar{\vartheta}$ and y_α. Taking into account the definition of the $\gamma_\pm$, where $y_o=\bar{y}$, we obtain

$$\nu=\frac{4\rho[2\bar{\vartheta}\cot(2\bar{\vartheta})-1]}{\rho+\mathrm{sech}(2\bar{y})},\quad \mu=\frac{4\rho\tanh(2\bar{y})}{\rho+\mathrm{sech}(2\bar{y})}\bar{\vartheta}.$$

Let us collect all the contributions, i.e., the terms same as for the Thirring soliton train [see Eqs. (29)–(31)] and those accounting for the γ terms, and write down the corresponding evolution equations for the parameters of the αth gap soliton. They read

$$\frac{d\vartheta_\alpha}{dt}=\sum_{\beta=\alpha\mp1} 2\sin^3(2\overline{\vartheta})e^{-|\Delta_{\alpha\beta}|}\sin\Psi_{\alpha\beta}, \tag{47}$$

$$\frac{dy_\alpha}{dt}=\sum_{\beta=\alpha\mp1} 4\sin^3(2\overline{\vartheta})e^{-|\Delta_{\alpha\beta}|}[\mathrm{sgn}(\Delta_{\alpha\beta})\cos\Psi_{\alpha\beta} - \kappa\sin\Psi_{\alpha\beta}], \tag{48}$$

$$\frac{d\delta_\alpha}{dt}=-\cos(2\vartheta_\alpha)+\mu\sin(2\vartheta)y_\alpha, \tag{49}$$

$$\frac{dx_\alpha}{dt}=(1+\nu)y_\alpha. \tag{50}$$

Here

$$\Delta_{\alpha\beta}=\sin(2\overline{\vartheta})(x_\beta-x_\alpha),\quad \Psi_{\alpha\beta}=\delta_\alpha-\delta_\beta.$$

It is easy to see that the averages $\overline{\vartheta}$ and $\overline{y}$ are not affected by Eqs. (47) and (48). Equations (47)–(50) are similar to those for the (almost quiescent) Thirring solitons, however, there are additional terms in the evolution equations for y_α, δ_α, and x_α.

Let us now derive the generalized complex Toda chain corresponding to Eqs. (47)–(50). As the derivation is quite similar to that for the *quiescent* Thirring solitons we will skip some details. As in the case of the massive Thirring model, introduce the modified time

$$\tau=\sin(2\overline{\vartheta})t,$$

an average phase

$$\overline{\delta}=-\cos(2\overline{\vartheta})t,$$

and the complex variable q_α for each soliton,

$$q_\alpha=-\sin(2\overline{\vartheta})x_\alpha-i(\delta_\alpha-\overline{\delta}+\alpha\pi)+2\alpha\ln[2\sin(2\overline{\vartheta})]. \tag{51}$$

Differentiating q_α and throwing away the second-order terms, one obtains

$$\frac{dq_\alpha}{d\tau}=-\{(1+\nu+i\mu)y_\alpha+i[2\vartheta_\alpha-2\overline{\vartheta}]\}.$$

Differentiation of this formula gives

$$\begin{aligned}\frac{d^2q_\alpha}{d\tau^2}=&-\sum_{\beta=\alpha\mp1} 4\,\mathrm{sgn}(\Delta_{\alpha\beta})\sin^2(2\overline{\vartheta})\\ &\times\exp\{\mathrm{sgn}(\Delta_{\alpha\beta})(-\Delta_{\alpha\beta}+i\Psi_{\alpha\beta})\}\\ &-(\nu+i\mu)\mathrm{Re}\sum_{\beta=\alpha\mp1} 4\,\mathrm{sgn}(\Delta_{\alpha\beta})\sin^2(2\overline{\vartheta})\\ &\times\exp\{\mathrm{sgn}(\Delta_{\alpha\beta})(-\Delta_{\alpha\beta}+i\Psi_{\alpha\beta})\}\\ &+\kappa(1+\nu+i\mu)\mathrm{Im}\sum_{\beta=\alpha\mp1} 4\,\mathrm{sgn}(\Delta_{\alpha\beta})\sin^2(2\overline{\vartheta})\\ &\times\exp\{\mathrm{sgn}(\Delta_{\alpha\beta})(-\Delta_{\alpha\beta}+i\Psi_{\alpha\beta})\},\end{aligned}$$

taking into account the numeration of the solitons in the train, which is given by $x_{\alpha+1}-x_\alpha>0$ or $\Delta_{\alpha\alpha+1}>0$, and the following identity:

$$4\sin^2(2\overline{\vartheta})\exp\{\pm(-\Delta_{\alpha\alpha\pm1}+i\Psi_{\alpha\alpha\pm1})\} =-\exp[\pm(q_{\alpha\pm1}-q_\alpha)].$$

We obtain a generalized complex Toda chain for the train of N well-separated gap solitons with nearly equal amplitudes and rapidities

$$\frac{d^2q_\alpha}{d\tau^2}=(1+A_\rho)(e^{q_{\alpha+1}-q_\alpha}-e^{q_\alpha-q_{\alpha-1}}) +B_\rho(e^{q^*_{\alpha+1}-q^*_\alpha}-e^{q^*_\alpha-q^*_{\alpha-1}}), \tag{52}$$

where A_ρ and B_ρ are ρ-dependent coefficients

$$A_\rho=\tfrac{1}{2}\{\nu-\kappa\mu+i[\kappa(1+\nu)+\mu]\},$$

$$B_\rho=\tfrac{1}{2}\{\nu+\kappa\mu-i[\kappa(1+\nu)-\mu]\}.$$

As usual, $\mathrm{Re}\{q_0\}=\infty$ and $\mathrm{Re}\{q_{N+1}\}=-\infty$.

Though in Eq. (52) and in the definition of q_α Eq. (51) we still have the variables τ, x_α, δ_α, and $\overline{\delta}$ defined through the coordinates x and t [see formula (39)], it is easy to reverse to the coordinates X and T of the optical gap system (1). Indeed, to this end one should use the transformation (39) (with $y_o=\overline{y}$) for the position x_α and the central phase δ_α of the gap soliton (the phase at $X=X_\alpha$),

$$x_\alpha=\cosh(\overline{y})[X_\alpha-\tanh(\overline{y})T], \tag{53}$$

$$\delta_\alpha=[-\cos(2\vartheta_\alpha)+\mu\sin(2\overline{\vartheta})y_\alpha]t=[-\cos(2\vartheta_\alpha) +\mu\sin(2\overline{\vartheta})y_\alpha]\cosh(\overline{y})[T-\tanh(\overline{y})X_\alpha]. \tag{54}$$

Also one must use the time transformation $dT=\cosh(\overline{y})dt$ in the definition of τ and the average phase $\overline{\delta}$,

$$\tau=\sin(2\overline{\vartheta})\mathrm{sech}(\overline{y})T,\quad \overline{\delta}=-\cos(2\overline{\vartheta})\mathrm{sech}(\overline{y})T.$$

Now it is evident that if Eq. (53) is used in the definition of q_α, Eq. (51), the term linear in T will not contribute to either the difference $q_\alpha-q_\beta$ or the second derivative of q_α, hence, it can be dropped. Further, notice that from the rhs of Eq. (54) only the term linear in T will appear in q_α, if one simply changes the time $t\to\mathrm{sech}(\overline{y})T$. Hence, the term proportional to X_α in Eq. (54) *must be subtracted* from the central phase δ_α. In doing so, one can neglect the difference between ϑ_α and $\overline{\vartheta}$ due to the inequalities (7) and that evolution of ϑ_α is of order $O(\epsilon)$ (i.e., we throw away the second-order terms from the second derivative of q_α). Thus we have arrived precisely at the quantity Q_α given by Eq. (17), where the shift of the soliton rapidities is taken into account, $y_\alpha \to \overline{y} + y_\alpha$. Therefore, the result of Sec. II is proven.

V. COMMENTS

The complex Toda chain model proves to be a universal model for the adiabatic description of the train interaction/propagation of solitons in nonlinear PDEs. Indeed, it was shown to describe the train propagation of pulses in the nonlinear PDEs of the whole NLS hierarchy [9] (i.e., the PDEs associated with the familiar Zakharov-Shabat spectral problem [52,53]). More recently, the complex Toda chain was derived for the soliton train of the modified NLS equation [10]. This PDE is associated with the quadratic bundle, also known as the Wadati-Konno-Ichikawa spectral problem [54].

In this paper, the complex Toda chain is shown to describe the soliton train propagation in the massive Thirring model. Note that, as it is mentioned in Ref. [10], the massive Thirring model is just another representative of the modified NLS hierarchy. Thus the complex Toda chain arises in the adiabatic description of the soliton trains in the hierarchy of nonlinear integrable PDEs associated with the quadratic bundle as well. This is in favor of the universality of the complex Toda chain.

In construction of the perturbation theory for the massive Thirring model we have used the associated Riemann-Hilbert problem [50]. The use of the Riemann-Hilbert problem allows one to develop the perturbation theory in a unified way for the entire hierarchy (see, for instance, Ref. [55], where this was done for the vector NLS hierarchy). Moreover, the perturbation-induced evolution equations for the spectral data have one and the same form for *all* integrable PDEs (one can compare the results of Refs. [56,57]). This gives a possibility to prove the universality of the complex Toda chain using the approach based on the Riemann-Hilbert problem. This is one of the directions for future work.

In view of recent experimental observation [34] of the multiple gap soliton formation in optical fibers with index grating, it is important to have an analytical approach for description of interaction of optical gap solitons. Some analytical results in this direction are already contained in Ref. [36], where the authors consider the *N*-soliton solutions to the optical gap system. In the present paper this approach is further developed and it is shown that the train interaction/propagation of *N*-gap solitons with nearly equal amplitudes and velocities is governed, in the adiabatic approximation, by a generalized complex Toda chain with *N* nodes.

Here we should mention that, due to nonintegrability of the optical gap system, the train of gap solitons may become unstable. Such instability can be the result of the soliton-radiation interaction and is beyond the adiabatic approximation. However, the gap soliton is stable against the effect of radiation if the soliton amplitude lies below the instability threshold (see for details Ref. [37]). For such values of the soliton amplitudes, the generalized complex Toda chain (18) can be applied.

Spurious instabilities of the complex Toda chain are also possible for some initial values, as was pointed out in Ref. [4]. Such instabilities do not correspond to instabilites of the soliton train propagation in the original PDE. However, recently, it was indicated [8] that the generalized complex Toda chains (for the soliton train propagation in the perturbed NLS equations) admit more stationary regimes of the train propagation than the (integrable) complex Toda chain does. Similar fact can be true for the gap soliton train propagation. This issue is important in view of applications of the gap solitons and it will be addressed in a future publication.

Though the generalized complex Toda chain is not integrable, it is a finite dimensional dynamical system and can be investigated by the standard techniques. Moreover, in accordance with discussion of Ref. [9], one can systematically include various additional perturbations of the optical gap system into the complex Toda chain. For instance, to address the issue of stability of the soliton train, one can study the interaction of the solitons in train with radiation waves by using the perturbation theory.

ACKNOWLEDGMENTS

The author gratefully acknowledges many stimulating discussions with Professor E. V. Doktorov and Professor V. S. Gerdjikov. Also, the author is indebted to Professor V. S. Gerdjikov for his critical reading of the manuscript. Some part of this work was done during the RCP 264 Conference (June 2000, Montpellier, France) and the author wishes to thank the organizers, Professor J.-G. Caputo and Professor P. Sabatier, for their support. This work was supported by the National Research Foundation of South Africa.

[1] J.M. Arnold, IEE Proc.-J: Optoelectron. **140**, 359 (1993).
[2] J.M. Arnold, in Proceedings of URSI Electromagnetic Theory Symposium, St. Peterburg (unpublished).
[3] V.S. Gerdjikov, D.J. Kaup, I.M. Uzunov, and E.G. Evstatiev, Phys. Rev. Lett. **77**, 3943 (1996).
[4] V.S. Gerdjikov, I.M. Uzunov, E.G. Evstatiev, and G.L. Diankov, Phys. Rev. E **55**, 6039 (1997).
[5] J.M. Arnold, J. Opt. Soc. Am. **15**, 1450 (1998).
[6] V.S. Gerdjikov, E.G. Evstatiev, and R.I. Ivanov, J. Phys. A **31**, 8221 (1998); **33**, 975 (2000).
[7] V.S. Gerdjikov, E.G. Evstatiev, D.J. Kaup, G.L. Diankov, and I.M. Uzunov, Phys. Lett. A **241**, 323 (1998).
[8] J.M. Arnold, Phys. Rev. E **60**, 979 (1999).
[9] V.S. Gerdjikov and I.M. Uzunov, Physica D **152-153**, 355 (2001).
[10] V.S. Gerdjikov, E.V. Doktorov, and J. Yang, Phys. Rev. E **64**, 056617 (2001).
[11] A.V. Mikhailov, in *Optical Solitons: Theoretical Challenges and Industrial Perspectives*, edited by V.E. Zakharov and S. Wabnitz (EDP Sciences, 1999), pp. 63-71.

[12] V.I. Karpman and V.V. Solov'ev, Physica D **3**, 487 (1981).
[13] M. Toda, *Theory of Nonlinear Lattices* (Springer-Verlag, Berlin, 1989).
[14] J. Moser, *Dynamical Systems*, *Theory and Applications*, Lecture Notes in Physics Vol. 38 (Springer-Verlag, Berlin, 1975), p. 467; Adv. Math. **16**, 197 (1975).
[15] W.E. Thirring, Ann. Phys. (N.Y.) **3**, 91 (1958).
[16] E.A. Kuznetsov and A.V. Mikhailov, Theor. Math. Phys. **30**, 193 (1977).
[17] C.M. de Sterke and J.E. Sipe, Phys. Rev. A **38**, 5149 (1988); **39**, 5163 (1989); **42**, 550 (1990).
[18] D.N. Christodoulides and R.I. Joseph, Phys. Rev. Lett. **62**, 1746 (1989).
[19] A.B. Aceves and S. Wabnitz, Phys. Lett. A **141**, 37 (1989).
[20] M. Romangoli, S. Trillo, and S. Wabnitz, Opt. Quantum Electron. **24**, S1237 (1992).
[21] M.J. Steel and C. Martijn de Sterke, Phys. Rev. A **48**, 1625 (1993).
[22] J. Feng and F.K. Kneubuhl, IEEE J. Quantum Electron. **29**, 590 (1993).
[23] N.G.R. Broderick, C. Martijn de Sterke, and J.E. Sipe, Opt. Commun. **113**, 118 (1994).
[24] M. Scalora, J.P. Dowling, C.M. Bowden, and M.J. Bloemer, Phys. Rev. Lett. **73**, 1368 (1994).
[25] N.G.R. Broderick and C. de Sterke, Phys. Rev. E **51**, 4978 (1995).
[26] Ze Cheng and G. Kurizki, Phys. Rev. Lett. **75**, 3430 (1995); Phys. Rev. A **54**, 3576 (1996).
[27] B.A. Malomed and R.S. Tasgal, Phys. Rev. E **49**, 5787 (1994).
[28] C. Martijn de Sterke and J.E. Sipe, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1994), Vol. XXXIII, Chap. III, pp. 203–260.
[29] G. Kurizki, A.E. Kozhekin, T. Opatrny, and B. Malomed, Prog. Opt. **42**, 93 (2001).
[30] R.H. Goodman, M.I. Weinstein, and P.J. Holmes, Nonlin. Sci. **11**, 123 (2001).
[31] B.J. Eggleton, R.E. Slusher, C.M. de Sterke, P.A. Krug, and J.E. Sipe, Phys. Rev. Lett. **76**, 1627 (1996).
[32] N.G.R. Broderick, D. Taverner, D.J. Richardson, M. Ibsen, and R.I. Laming, Phys. Rev. Lett. **79**, 4566 (1997).
[33] D. Taverner, N.G.R. Broderick, D.J. Richardson, M. Ibsen, and R.I. Laming, Opt. Lett. **23**, 259 (1998).
[34] D. Taverner, N.G.R. Broderick, D.J. Richardson, R.I. Laming, and M. Ibsen, Opt. Lett. **23**, 328 (1998).
[35] M. Romangoli, S. Trillo, and S. Wabnitz, Opt. Quantum Electron. **24**, S1237 (1992).
[36] A.B. Acevez and S. Wabnitz, in *Nonlinear Processes in Physics*, edited by A.S. Fokas, D.J. Kaup, A.C. Newell, and V.E. Zakharov (Springer, Berlin, 1991).
[37] I.V. Barashenkov, D.E. Pelinovsky, and E.V. Zemlyanaya, Phys. Rev. Lett. **80**, 5117 (1998).
[38] A. De Rossi, C. Conti, and S. Trillo, Phys. Rev. Lett. **81**, 85 (1998).
[39] Yu.S. Kivshar and N. Flytzanis, Phys. Rev. A **46**, 7972 (1992).
[40] Yu.S. Kivshar, Phys. Rev. Lett. **70**, 3055 (1993).
[41] C. Martijn de Sterke, Phys. Rev. E **48**, 4136 (1993).
[42] A.V. Buryak and Yu.S. Kivshar, Opt. Lett. **19**, 1612 (1994).
[43] Yu.S. Kivshar, Phys. Rev. E **51**, 1613 (1995).
[44] Yu.S. Kivshar, O.A. Chubykalo, O.V. Usatenko, and D.V. Grinyoff, Int. J. Mod. Phys. B **9**, 2963 (1995).
[45] C. Conti, S. Trillo, and G. Assanto, Phys. Rev. Lett. **78**, 2341 (1997).
[46] T. Peschel, U. Peschel, F. Lederer, and B.A. Malomed, Phys. Rev. E **55**, 4730 (1997).
[47] H. He and P.D. Drummond, Phys. Rev. Lett. **78**, 4311 (1997).
[48] C. Conti, S. Trillo, and G. Assanto, Phys. Rev. E **57**, R1251 (1998).
[49] C. Conti, S. Trillo, and G. Assanto, Opt. Lett. **23**, 334 (1998).
[50] V.S. Shchesnovich and I.V. Barashenkov (unpublished).
[51] D.J. Kaup and T.I. Lakoba, J. Math. Phys. **37**, 308 (1996).
[52] V.E. Zakharov and A.B. Shabat, Sov. Phys. JETP **34**, 62 (1972).
[53] S.P. Novikov, S.V. Manakov, L.P. Pitaevski, and V.E. Zakharov, *Theory of Solitons the Inverse Scattering Method* (Consultants Bureau, New York, 1984).
[54] M. Wadati, K. Konno, and Y.H. Ichikawa, J. Phys. Soc. Jpn. **46**, 1965 (1979).
[55] V.S. Shchesnovich, J. Math. Phys. **43**, 1460 (2002).
[56] V.S. Shchesnovich and E.V. Doktorov, Phys. Rev. E **55**, 7626 (1997).
[57] V.S. Shchesnovich and E.V. Doktorov, Physica D **129**, 115 (1999).